Mathematical Modeling: Transitions between the Real World and the Mathematical Model

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Applications in engineering, science and technology within undergraduate programmes can be difficult for students to understand. In this paper, new results are presented which go some way to demonstrate and explain the problems faced by students in linking mathematical models to real-world applications. The study is based on student responses to multiple-choice questionnaires on mathematical modelling problems, student reflective questionnaires and subsequent interviews. The processes used by students to solve such problems are classified and links between these process levels, credit and seven other behavioural descriptors are discussed.

1. Mathematical models and modelling

Students of engineering, science and technology are familiar with mathematical models and their use for their own conceptual development within the discipline. Mathematical modelling, which involves moving from a real-world situation to a model, working with that model and using it to understand and to develop or solve the real-world problems, has long been a feature of courses in engineering, science and technology as well as in mathematics. The activity is recognized in curricula through investigations and projects, undertaken either as a group or individually and, while the focus is on the modelling itself, there is also the potential to enhance the performance in mathematics of students generally. Indeed, Matos [1] reinforces this view:

> .. .mathematical modelling.. .[is].. .an activity where students give meaning to ideas, problems [and] mathematical and non-mathematical concepts. [1, p. 26]

This broader emphasis, inclusive of mathematics and beyond mathematics, is particularly important in engineering, science and technology where transitions between real-world problems and the model are the substance of the discipline. The processes by which students build mathematical ideas and structures within scientific disciplines are complex, as are the teaching and learning paradigms within which they work. In teaching, particularly in engineering, science and technology, the presentation of models and their critical analysis has been prevalent, although the increasing use of projects and investigations has led to the definition and examination of the processes through which the successful modeller passes. The cyclic process can be said to consist of several stages:

> real world problem statement; formulating a model; solving mathematics; interpreting outcomes; evaluating a solution; refining a model; real world problem statement... [2],

and a further *reporting* stage occurring after evaluating a solution.

It can be noted here that, while written and oral reports in various forms are an established part of the *scientific method* adopted in engineering, science and technology, they also provide opportunities to consider communication in mathematics and communicating mathematics more directly and in particular, through the activity of mathematical modelling, communicating mathematical meaning [3].

Of course, the solving mathematics stage is central to student success in mathematical modelling though, unfortunately, students can be found wanting in this area. Anderson *et al*. [4] found that even final year mathematics undergraduates tended to memory dump in attempting to solve examination questions rather than retaining and building upon a strong and coherent structure in mathematics. From an extensive study of problem-solving, Galbraith and Haines [5] describe a hierarchy of procedural and conceptual skills. They reported on mechanical, interpretive and constructive problem-solving skills where the relative degree of success on these three different types of problem is

$$\text{mechanical} > \text{interpretive} > \text{constructive}$$

and demonstrated the validity of this taxonomy. The research of Anderson *et al*. [4] and Galbraith and Haines [5] is concerned with the mathematical structures developed and deployed by students in solving mathematics problems. Their research shows clear behavioural aspects linking the two, which are most apparent in the deployment of mechanical skills. On the other hand, Kent and Noss [6] examined perceptions of engineering undergraduates towards mathematics in a study of the use of technology as a bridge between mathematics and engineering. In this environment, they suggest that increasingly the teaching of mathematics to engineers will be a succession of recipes wrapped in a computational dressing with an implication that less emphasis may be placed on interpretative and conceptual skills [5] unless these are targeted in teaching. The results reported by these authors [4–6] capture core problem-solving behaviours of students, which impact on mathematical modelling competencies at the solving mathematics stage of the modelling cycle.

It is, however, the interface between the real world problem and the mathematical model that presents difficulties for students. The transition from the real world to the mathematical model and (having considered the model and its solution(s)), the transition from that solution back to the real-world problem, might seem to be straightforward to the expert modeller but for a student new to mathematical modelling, this is not necessarily so.

2. Expertise in problem solving and becoming an expert modeller

Successful mathematical modelling involves an ability to move between the real world and the mathematical world, bearing both in mind. The modeller needs to consider the real-world problem and decide how to mathematise it, deciding which aspects of the real-world problem are relevant and which not—a process of abstraction—and deciding what mathematical principles and techniques to bring to bear, even when technology is used to apply them [6]. The solution also needs to be checked against the reality provided by the engineering, scientific or technology context and modified if necessary. These processes are demonstrably difficult for students in a variety of countries who are new to modelling. Haines and Crouch [7] and Haines *et al*. [8] report such difficulties among new undergraduates in the UK, Ikeda and Stephens [9, pp. 381, 382] recorded problems among Japanese students engaged in relatively open modelling tasks and problems were also found by Klymchuk and Zverkova [10] in a study of 500 students from 14 universities in Australia, Finland, France, New Zealand, Russia, South Africa, Spain, Ukraine and the UK.

Why is it that students of engineering science, technology and allied subjects find it difficult to move freely between the real world and the mathematical world, when by their own choice of applied discipline one might have expected strong engagement in modelling or pseudo-modelling tasks? Formal education usually leads to a greater ability to abstract, because practice is provided in thinking about topics in a decontextualised way [11, p. 473], but there are associated problems in linking this decontextualized knowledge back to the real world. One reason might be that in abstraction and decontextualization, the rich connections that provide the motivation for the subject discipline are lost. Nunes and Bryant [12, pp. 234–348] in a review of research from various countries, conclude that the mathematical capabilities of children are greatly influenced by whether they are in a real world or a classroom context. They note difficulties in transferring skills and in knowledge from one context to the other, and particularly in bringing understanding from the real world into the classroom, where children may define their competence of maths as their ability to use socially approved solutions. Such solutions are very likely to focus on the results of abstractions mentioned above. A similar effect was found by Christiansen [13] among 10th grade Swedish students, on a mathematical modelling course, where the classroom context made it difficult for students to make successful links between the real world and the mathematical world, instead seeing the problem as one of finding out what procedures the teacher wanted them to apply. Although the experiences described above [12, 13] are pre-university, they do impact on how undergraduate students tackle mathematical modelling when they eventually reach university.

Many students' difficulties may be due to their being comparatively new to mathematical modelling. Such novices, in a variety of fields, tend to perform poorly compared to experts, as may be expected. This seems to be due to novices possessing a much smaller and more poorly structured knowledge base, making it difficult for them to know which information is relevant, what type of problem they are dealing with and to know which techniques and procedures to apply, while experts generally have the experience and knowledge to do this successfully [14]. For instance novices tend to apply familiar procedures to algebra problems rather than relevant ones and be unable to understand the relationships among the variables unless these are made explicit [15]. Novices, unlike experts, cannot readily distinguish the relevant from the irrelevant as they have difficulty accessing and linking knowledge [14, 16]. Even if they can access relevant knowledge, they find it difficult to keep it all in mind [17]. The process of abstraction (identifying relevant problem features and choosing and testing a restricted set of hypotheses to explain them) is particularly difficult for novices [16, 18, 19].

Novices have difficulty recognizing types of problem and therefore in accessing associated solutions. With physics problems, for example, they tend to rely on the words and surface features in the problem statement and categorize problems according to these surface features rather than underlying principles, as experts do [20]. They spend less time planning and tend to approach the problem differently to experts. When solving physics problems, novices look for givens and unknowns and then immediately generate equations containing the unknown, working backwards from the goal in a way that is often less productive, especially with more complex problems [17, 20, 21]. When novices do appear, like experts, to have a tendency to work forwards from the problem statement, as when solving mechanics problems, they are not very competent at it [22]. When solving mathematics and physics problems, novices tend to press on regardless, not realizing when they have misunderstood, whereas experts tend to know when they have started on an unproductive path and backtrack [23, 24].

Many of these novice behavioural descriptors are commonly recognized in new undergraduates whose degrees require them to do mathematical modelling. Some of the expert descriptors are recognized among more experienced undergraduates in later years and even more in research students [8].

3. Modelling behaviours among new undergraduates

So that student achievement can be recognized, it is necessary to understand the developmental processes through which the learner passes in moving from novice behaviour to that of an expert. To investigate aspects of novice modelling behaviour, within a wider study in mathematical modelling, Haines *et al*. [8] used multiple-choice questions developed in analogue pairs, a student reflective questionnaire on individual answers to four unrelated multiple-choice questions and an interview by a tutor following the completion of the four multiple-choice questions and the reflective questionnaire to try to understand these processes. The reflective questionnaire and subsequent interview enabled a classification of the processes used by undergraduates (novices) and engineering research students (experts) in solving the multiple-choice questions at three levels:

Level a where there was clear evidence that they took into account the relationship between the mathematical world and the real world input to the model;

Level b where there was limited evidence that they took into account the relationship between the mathematical world and the real world input to the model, such as

(i) mentioned in interview that they had thought about the model, but the reflective questionnaire offers little evidence that this had been done, or

(ii) had obviously thought about the model, confirmed in the reflective questionnaire or in the interview, but lacked knowledge of the real world and/or mathematics to solve the problem effectively;

Level c (i) no-evidence that the relationship between the mathematical world and the real-world input to the model had been taken into account nor that a modelling perspective had been adopted, or

(ii) the problem had been looked at simply in real-world terms, or entirely in terms of reasoning or mathematics (according its position in the modelling cycle) without reference to the needs of the model nor to the interface between the mathematics and the real world.

An analysis of student responses among 25 novices, using this classification, demonstrated clearly that these students have difficulty in linking the real world and the mathematical model (Table 1) [8].

Questions	a	b	c	location in modelling cycle
1,2	0.18	0.18	0.64	real world to model
3,4	0.00	0.18	0.82	real world to model
5,6	0.00	0.33	0.67	specifying model
7,8	0.58	0.17	0.25	specifying variables
9,10	0.91	0.09	0.00	constructing equations
11,12	0.33	0.58	0.08	maths to real world
13,14	0.91	0.09	0.00	graphs to real world
15,16	0.58	0.08	0.33	using mathematics

Table 1. The proportion of student responses classified a, b and c (*n* ¼ 11 or 12). Proportions > 0.5 highlighted [8].

This paper reports on further analysis, for the research tools previously employed, enabling a consideration of student responses to nine descriptors of student behaviour:

1. the process (a, b, c) used by the students;
2. the credit that their chosen answer attracted using partial credit model (0, 1, 2);
3. whether the multiple-choice questions were easy or hard to understand;
4. whether the students regarded the multiple-choice questions as real world problems;
5. whether they found the multiple-choice questions interesting;
6. whether or not the students considered that the multiple-choice questions were located in mathematics as a discipline;
7. the time that students took to choose their answer from the five available options;
8. the ease with which they made that choice;
9. the confidence that students had in their chosen answer being a good one.

Student responses on these nine descriptors were analysed from three points of view: (i) an overall perspective; (ii) an individual student perspective, and (iii) a modelling cycle perspective. The correlation analysis led to the following results:

An overall perspective

In this section the 92 individual responses made by 23 students are considered. There is a connection between the process used by the student to solve the problem and the marks obtained for the answer. It is likely therefore, that encouraging behaviour at higher process levels will lead to higher credit (correlation 0.45). It was also found that there is a moderate correlation (0.41) between the process level and the students' view as to whether the problem was located within mathematics as a discipline, but the nature of that link (if it exists) is far from clear. That students think the multiple choice questions are easy to understand did not appear to influence credit, perhaps reflecting that novices often think they have understood a problem when they have not entirely done so. Thinking the question was easy to understand did, however, correlate with their confidence in their chosen answer (0.46) and not unexpectedly with the ease with which they were able to choose their answer (0.49). A link between the ease with which they were able to choose an answer, and confidence in that answer, is also confirmed (0.49). All these latter behaviours suggest that, given contextual differences in science and engineering and diversities amongst students, it is imperative that they can understand and identify with the problems.

A student perspective

In this section the consolidated responses for each of our 23 students on four multiple-choice questions are considered. The process used by the student to solve their four problems and the marks obtained for that answer are linked with the implication that behaviour at higher process levels will lead to higher credit (0.40). Such process behaviour instils confidence in their chosen answers (0.55) and the ease with which those answers were chosen (0.62). Among individual students there is little evidence of a link between the process level and the students' view as to whether the problems were located within mathematics as a discipline, lending support to a view that the nature of such connections is far from clear. Again, that students find the multiple choice questions easy to understand did not appear to influence credit, but it did correlate with the ease with which they were able to choose their answer (0.50). The link between the ease with which they were able to choose their four answers, and confidence in those answers, is strong (0.56). Here there is some evidence that it is harder for students to succeed when faced with real-world problems, such as are common in science and engineering. There is a weak but significant negative correlation between obtained credit and the students' perception that the questions are real world problems (−0.36), so that when faced with practical real world problems, students are less likely to succeed. There was further evidence of anxiety among students when faced with problems that they regard as located in mathematics. This is demonstrated in weak but negative correlations between 'in mathematics' and six of the remaining eight descriptors. Generally speaking, it appears that some expert behaviours are not yet fully developed, indicating a weak knowledge base and a lack of experience in abstraction.

A modelling cycle perspective

Real world to mathematical world. In this section the consolidated responses to questions 1–4, which were all concerned with moving from the real world to the mathematical world are considered. There is a connection between the process levels at which students operate and credit obtained (0.53); however, this simply indicates that behaviours using process 'a' (taking into account the relationship between the mathematical world and the real-world input) will result in higher credit that than those using process 'b'. The process classification (Table 1) requires continual referencing and reinforcement between the real-world problem and the mathematical model. It is clear that some of the students showing some evidence of expert behaviours can indeed move successfully from the real world to the mathematical model, but most cannot yet manage it; indeed there is evidence that higher process levels are related to a perception that the problem is not in the real world (— 0.56). Higher process levels might also link with the ease with which the answer was chosen (0.39), although many students have misplaced confidence in their answers; there is a weak negative correlation between credit and students' confidence in their chosen answer (– 0.24). This could suggest that most students cannot keep the needs of the real world and the mathematical model in mind at once. There are, in this part of the modelling cycle especially, stronger links between whether or not the questions were easy to understand and confidence in the chosen answer (0.69), the ease with which the answer was chosen (0.56) and the time taken to choose that answer (0.54). There is also some evidence of a relationship between the time taken to choose an answer and credit (0.40), although reasons for this are not apparent. One can understand a link between time taken to choose an answer, the ease of choosing that answer (0.52) and confidence in that answer (0.46), and also between the ease of choosing an answer and confidence in that answer (0.55). Once more, expert behaviours are observed in some students, but this is hampered by lack of knowledge and inexperience in abstraction at this stage of the modelling cycle.

Formulating and working with a mathematical model. In this section the consolidated responses to questions 5–10, which were all concerned with formulating and working with a mathematical model, are considered. In this part of the modelling cycle students are in more familiar territory working within mathematics and links between process and credit are strong (0.51). However, as in the early part of the modelling cycle, the achievement of credit, and student confidence are not linked strongly. In fact, in this area, confidence appears to be related simply to whether or not the students found the question easy to understand (0.57) and to the ease with which they were able to choose an answer (0.54). There is evidence, however, that credit is linked to the time taken to choose an answer (0.43), which could be a manifestation of process and the skills used to solve the problems. Higher skills levels and increasing certainty about method and methodology would lead to more straightforward choices among the options given and is supported by the correlation between time to choose an answer and ease of making that choice (0.51).

Mathematical model to the real world. In this section the consolidated responses to questions 11–16, which were all concerned with moving from the mathematical model to the real world, are considered. At the end of the modelling cycle, higher processes are more in evidence but this does not strongly correlate with credit (0.31) indicating weaker links here than in other parts of the cycle. The stronger links between process and a student's interest in the problem (0.49) and between process and a student's confidence in their chosen answer (0.53) provide evidence of students engaging well with the problem which is always important behaviour, especially in science and engineering. There is some evidence that although students found the problems easy to understand this did not necessarily lead to credit (— 0.31) although there is again a link to the ease with which an answer was chosen (0.46).

4. Linking the mathematical world and the real world

In trying to understand how students acquire mathematical modelling skills, appropriate to engineering, science and technology, several consistent behaviours are identified through our assessment and evaluation strategies. Some differences in application have revealed common outcomes. This research suggests that students are weak at linking the mathematical world and the real world, thus supporting a view that students need much stronger experiences in building real world mathematical world connections. The weakness in students linking real world and mathematics which has been reported in other studies [9, 10], can be addressed straightforwardly. It is not difficult to provide experiences in this transition; one example, among many, is given by Watson [25] who shows how probability may be introduced and developed in Australian schools through newspapers and the media. It is clear also that teaching and learning styles need to focus more strongly on abstraction and the formulation of the mathematical model [10]. In a review of existing research on the teaching and learning of applications and modelling, Niss [26] found that students' capability on applications and modelling tasks was likely to be influenced by the teaching approach, the context and situation in which the mathematical modelling task is embedded, student and teacher motivation, engagement with and attitudes towards modelling work, and the amount of effortful practice and experience students had on modelling tasks.

Since groupwork and projects are a common feature of undergraduate programmes in engineering science and technology, it is appropriate to note the effect of groupwork and its organization on mathematical tasks. In an interesting study of 36 Year 5 children in Queensland, Cooper *et al*. [27] show that groupwork is effective for developing expertise in mathematical problem solving and also how defined roles for members of the group affect problem-solving performance. Of course, aspects of group behaviour also affect performance as Ikeda and Stephens [9] show among their Japanese students, where discussion of the tasks at hand within the group resulted in stronger achievement for the group in modelling tasks. Some of the new research results reported here are undoubtedly affected by the context chosen for the multiple-choice questions and both the real world experiences of the students and the mathematical knowledge held by the students. Solving problems in mathematical modelling requires a construction of meaning within modelling and acknowledging differing social contexts no less important than that described by Nathan [28] when reporting on the difficulties of engaging Maori children in mathematics when the language of the Maori does not readily lend itself to mathematical concepts developed for the most part in Europe.

The development of expertise in a variety of fields, including an appropriately structured and organized knowledge base appears to take an extended time to develop and needs a basis of appropriate instruction with a large amount of motivated practice at suitable tasks with appropriate learning goals [29]. Students need to be engaged with the tasks and receive appropriately timed feedback [30–32].

The move to expertise involves building on this basis to self-develop appropriate knowledge and skills [29].

In the development of mathematical modelling skills, there is a need for practice on more open tasks that are not part-mathematized already, in a realistic context. Students' knowledge of relevant aspects of the real world also influences their ability to model [33]. Klymchuk and Zverkova [10] found that students across nine countries all tended to feel that they found moving from the real to the mathematical world difficult because they lacked such practice in application tasks. Teachers need to help students to see coherent links between the real-world context and the abstracted formalisms [34, p. 60]. It would also be helpful to analyse what subtasks are involved in the process of moving between the real and the mathematical world in the modelling process and which of these cause the most difficulty [26].

The demands of assessment may influence student motivation and attitude. Klymchuk and Zverkova [10] also found that while most students found application problems more interesting, more than half of the students preferred tests to consist of pure mathematics problems because they were easier to pass, and with application problems they had difficulty moving from the word problem to mathematical language. Interestingly at a few (unspecified) universities almost all the students actually preferred application problems in tests despite considering them harder. This suggests the need to investigate the possibility of variation in motivation to practise aspects of modelling skills perceived as more difficult (involving moving between the real world and the mathematical world), perhaps depending on the cultural and educational environment, including assessment practices.

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